Lecture 8 - More on quasi-coherent sheaves and stacks

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$\boxed{\text{prop:}}$ A for a morphism $X_\bullet \to Y_\bullet$, $\exists$ pullback functor $QCoh(Y_\bullet) \to QCoh(X_\bullet)$, which is an equivalence if f is morita, and natural trans. gives iso. of functors

The cleanest way to think of this is given a diagram of schemes, what is a Quasi-coherent sheaf on that diagram (Cartesian)

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

arrow $\mathcal{E} \to \mathcal{F}$ is an isomorphism. $f^*\mathcal{F} \xrightarrow{\cong} \mathcal{E}$

<u>Claim:</u> A quasicoherent sheaf is the same as a Cartesian quasicoherent sheaf on the diagram

$$X_2 \rightrightarrows\!\!\!\to X_1 \rightrightarrows X_0 \circlearrowleft$$

<u>Proof of prop:</u>

Step 1) Banal groupoid for fpqf map

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↳ can reduce to case of affine fppf maps, statement about ring maps

2) for general groupoids, look at

$$X_\bullet \longrightarrow Y_\bullet$$

$$\begin{tikzcd} W_{00} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_1 \arrow[r, "t"] \arrow[d, "s"] & Y_0 \\ X_0 \arrow[r] & Y_0 & \end{tikzcd}$$

motivation

$$\left(\begin{tikzcd} X_0 \arrow[r] \arrow[dr, phantom, "\llcorner"] & \mathfrak{X} \\ W_{00} \arrow[u] \arrow[r] & Y_0 \arrow[u] \end{tikzcd}\right)$$

3) Use the fact that

$$QCoh(X_\bullet) \xrightarrow{\cong} QCoh(W_{\bullet\bullet}) \xleftarrow{\cong} QCoh(Y_\bullet)$$

∎

Thus we think of $QCoh(X_\bullet)$ as indep. of presentation.

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Already, we "secretly" used the notion of "fibered" categories: <u>definition</u>: existence of Cartesian arrows

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \\ Sch \end{tikzcd} \qquad \begin{tikzcd} \zeta \arrow[r, dashed] \arrow[rr, bend left] \arrow[d, mapsto] & \xi \arrow[r, "\theta"'] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ p(\zeta) \arrow[r] & p(\xi) \arrow[r] & p(\eta) \end{tikzcd}$$

We saw the example of $QCoh = \{(X, \mathcal{E})\}$

Think of this as a functor from schemes to categories

$$QCoh(U) := p^{-1}(U) \subset QCoh$$

Given a diagram of schemes, a quasi-coh. sheaf on that diagram is a <u>Cartesian</u> section

$$\begin{tikzcd} & QCoh \arrow[d] \\ D \arrow[r] \arrow[ur, dashed, "\mathcal{E}"] & Sch \end{tikzcd}$$

This leads to the correct notion of a stack (modeled after the stack of q-coh. sheaves).

Def: A <u>stack</u> is a category fibered in groupoids such that $\forall$ banal groupoids associated to an etale cover $U \longrightarrow X$ of schemes,

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U_\bullet)$$

is an equivalence.

(Vistoli)

4.1.3. Fibered categories with descent.

DEFINITION 4.6. Let $\mathcal{F} \to \mathcal{C}$ be a fibered category on a site $\mathcal{C}$.

(i) $\mathcal{F}$ is a *prestack* over $\mathcal{C}$ if for each covering $\{U_i \to U\}$ in $\mathcal{C}$, the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is fully faithful.

(ii) $\mathcal{F}$ is a *stack* over $\mathcal{C}$ if for each covering $\{U_i \to U\}$ in $\mathcal{C}$, the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is an equivalence of categories.

<u>Ex:</u> the stack $QCoh^{cart}$

<u>Ex:</u> Stack of G-bundles

<u>Ex:</u> Quotient stacks

$\hookrightarrow$ these are <u>algebraic</u>,

To define this, some notions

1) base preserving functors
2) 2-fiber product

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3) 2-yoneda lemma

2-YONEDA LEMMA. *The two functors above define an equivalence of categories*

$$\mathrm{Hom}_{\mathcal{C}}((\mathcal{C}/X), \mathcal{F}) \simeq \mathcal{F}(X).$$

4) definition of representable maps